

# Relativistic $r$ -modes and shear viscosity

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**Abstract.** We derive the relativistic equations for stellar perturbations, including in a consistent way shear viscosity in the stress-energy tensor, and we numerically integrate our equations in the case of large viscosity. We consider the slow rotation approximation, and we neglect the coupling between polar and axial perturbations. In our approach, the frequency and damping time of the emitted gravitational radiation are directly obtained. We find that, approaching the inviscid limit from the finite viscosity case, the continuous spectrum is regularized. Constant density stars, polytropic stars, and stars with realistic equations of state are considered. In the case of constant density stars and polytropic stars, our results for the viscous damping times agree, within a factor two, with the usual estimates obtained by using the eigenfunctions of the inviscid limit. For realistic neutron stars, our numerical results give viscous damping times with the same dependence on mass and radius as previously estimated, but systematically larger of about 60%.

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## INTRODUCTION

Rotating neutron stars are a promising source of gravitational waves. Indeed, their oscillations can become unstable, converting their rotational energy into gravitational radiation. It is of particular interest the instability of the  $r$ -modes, which are oscillation modes of the star associated with the Coriolis force, and can become unstable at any rotation rate of the star [1].

It has been suggested that  $r$ -mode instability could produce waves strong enough to be detected by LIGO and VIRGO[2]. Furthermore, these instabilities are thought to play an important role in astrophysics. For instance, they may be relevant in spinning-down newly born neutron stars (and quark stars, if they actually exist) [3, 4], and could be responsible for preventing the spin-up of accreting neutron stars in low mass  $X$ -ray binaries [5, 6].

In spite of the extensive literature dedicated to the study of the problem, our present understanding of  $r$ -mode instability of compact stars is still incomplete, with many controversial issues.

It has been pointed out [7, 8, 9] that, working at first order in perturbation theory and neglecting the coupling between different harmonics, leads to the existence of a continuous spectrum and makes doubtful the existence of the modes. However, non-perturbative numerical simulations [10, 11] seem to indicate that  $r$ -modes do exist, so that the continuous spectrum may be interpreted as an artifact due to an inconsistency of the perturbative expansion when  $\sigma \sim \Omega$ . With this motivation, more sophisticated methods to solve the eigenvalue problem have been developed [12, 13, 14].

Another interesting issue, which up to now has not been fully understood, is the role of viscosity in  $r$ -mode oscillations. It is believed that bulk/shear viscosity limit the instability at high/low temperatures, respectively. But a complete understanding of this mechanism is still missing. Furthermore, it would be interesting to understand if the inclusion of viscosity does affect the existence of the continuous spectrum, as suggested for example in [14].

Here we report the results of [15], where we have studied the effect of introducing shear viscosity to the  $r$ -mode oscillations frequencies and damping times. We have introduced shear viscosity from the beginning in the stress-energy tensor, solving perturbatively the Einstein’s equations; in this way we have been able to evaluate consistently both the frequency and the damping time of the mode, for the first time. A similar self-consistent inclusion of the heat transfer corrections has recently been done in [16].

Furthermore, we have found that by including a small amount of viscosity we are able to regularize the continuous spectrum.

## FORMULATION OF THE PROBLEM

We have considered a star rotating uniformly with angular velocity  $\Omega$ . At first order in  $\Omega$  (or, more precisely, at first order in the rotational parameter  $\varepsilon \equiv \Omega/\Omega_N$  with  $\Omega_N \equiv \sqrt{2M/R^3}$ ), the stationary background is described by the metric [17, 18]

$$ds^2 = g_{\mu\nu}^{(0)} dx^\mu dx^\nu = -e^{v(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\phi^2) - 2r^2 \omega(r) \sin^2 \vartheta d\phi dt, \quad (1)$$

where  $\omega(r)$  represents the dragging of the inertial frames. It corresponds to the angular velocity of a local ZAMO (zero angular momentum observer), with respect to an observer at rest at infinity. The 4-velocity of the fluid is simply  $u^{(0)\mu} = (e^{-v/2}, 0, 0, \Omega e^{-v/2})$ , and the stress-energy tensor is

$$T_{\mu\nu}^{(0)} = (\rho + p)u_\mu^{(0)}u_\nu^{(0)} + pg_{\mu\nu}^{(0)}. \quad (2)$$

We have assumed that viscosity does not affect the stationary axisymmetric background, because the shear tensor do vanish there. Therefore, in this regime the Einstein's equations reduce to the standard TOV (Tolman-Oppenheimer-Volkoff) equations plus a supplementary equation for the frame dragging  $\omega(r)$ :

$$\bar{\omega}_{,rr} - \left(4\pi(\rho + p)e^\lambda r - \frac{4}{r}\right) \bar{\omega}_{,r} - 16\pi(\rho + p)e^\lambda \bar{\omega} = 0 \quad (3)$$

where  $\bar{\omega}(r) \equiv \Omega - \omega(r)$ .

Taking into account viscosity, the stress-energy tensor has the form (see for instance [19]):

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} - 2\eta\sigma_{\mu\nu} - 2\zeta g_{\mu\nu}u^\alpha{}_{;\alpha}. \quad (4)$$

Here  $\eta$ ,  $\zeta$  are the shear and bulk viscosity coefficients (see [20] for a discussion on the meaning of these coefficients), and

$$\sigma_{\mu\nu} \equiv \frac{1}{2}g^{\rho\lambda} (u_{\mu;\rho}P_{\nu\lambda} + u_{\nu;\rho}P_{\mu\lambda}) - \frac{1}{3}u^\rho{}_{;\rho}P_{\mu\nu} \quad (5)$$

with  $P_{\mu\nu}$  being the projector onto the subspace orthogonal to  $u_\mu$

$$P_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu. \quad (6)$$

We have focused our investigation to  $r$ -modes, which are perturbations with axial symmetry. Therefore, we have considered axial perturbations of the metric (1), expanded in tensor spherical harmonics, and we have solved the Einstein's equations, linearized around the background, for such perturbations:

$$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu}. \quad (7)$$

This approach allows to determine the complex frequency of the quasi-normal modes of the star (and then, of the emitted gravitational radiation)

$$\sigma = 2\pi\nu + \frac{i}{\tau} \quad (8)$$

where  $\nu$  is the real frequency of the mode, and  $\tau$  is its damping time.

We have neglected the coupling between perturbations with harmonic indexes  $l$  and  $l \pm 1$ , as in [9, 21]. Under this approximation, bulk viscosity is not coupled to axial perturbations, because it enters into the equations only through the axial-polar  $l \leftrightarrow l \pm 1$  couplings. Thus, we only have studied the effects of shear viscosity, leaving the investigation of the effects of bulk viscosity for future work.

We also have considered the Cowling and the Newtonian limits of our equations, in order to make a detailed comparison with previous works.

Because of numerical problems, we cannot integrate our system of equations for  $\eta \lesssim 10^{-6} \text{ km}^{-1}$ .<sup>1</sup>

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<sup>1</sup> Usually the typical viscosity at  $T = 10^7 \text{ K}$  and  $\rho = 10^{15} \text{ g/cm}^3$  is  $\eta \sim 10^{23} \text{ g/cm/s} = 2.4 \cdot 10^{-11} \text{ km}^{-1}$ .

## RESULTS

Here we discuss the results of the numerical integration of our perturbative equations, carried out in [15]. We have studied different kinds of stellar models: constant density stars, polytropic stars and realistic neutron stars. In each case, we have compared the Newtonian limit, the Cowling approximation, and the relativistic calculation, in order to establish the qualitative differences between the three approaches and to compare to previous literature on the subject.

### Homogeneous stars

In the case of constant density stars, it has been shown that the frequencies of the  $r$ -modes lay outside the continuous spectrum (see e.g. [22] for the full GR results). We consider a uniform density star with a central energy density of  $10^{15} \text{ g/cm}^3$ , mass of  $1.086 M_\odot$  and radius of 8.02 km ( $M/R = 0.2$ ), with the rotational parameter  $\varepsilon = 0.3$ . In Fig. 1 we show the  $r$ -mode frequency versus the viscosity parameter  $\eta$ , assumed to be constant throughout the star. The Newtonian, Cowling, and General Relativistic results are shown by dotted, dashed, and solid lines, respectively. The Newtonian and GR frequencies in the inviscid limit (1064 Hz and 1144 Hz) are indicated as dotted and solid horizontal lines, while the shadowed region indicates the continuous spectrum. As shown in Fig. 1, as the viscosity decreases the inviscid Newtonian and GR results are recovered, while the frequency in the Cowling approximation falls inside the continuous spectrum.

Since the convergence to the inviscid limit is reached for  $\eta \approx 10^{-5} \text{ km}^{-1}$ , the mode frequency in the Cowling approximation can be found extrapolating the dashed line for  $\eta \rightarrow 0$ , and the corresponding value is 1244 Hz, about a ten percent larger than the GR value. It should be stressed that the mode frequency in the Cowling approximation in the inviscid limit had never been possible before for perfect fluids.

We stress that, as shown in Fig. 2, the Newtonian, Cowling and GR damping times coincide. All damping times differ for less than 20% and show a  $1/\eta$  behaviour, as expected from previous estimates in the literature. In particular, we can compare our results with the analytic formula of [20], resulting from Newtonian estimates of the dissipative time scale of the shear viscosity

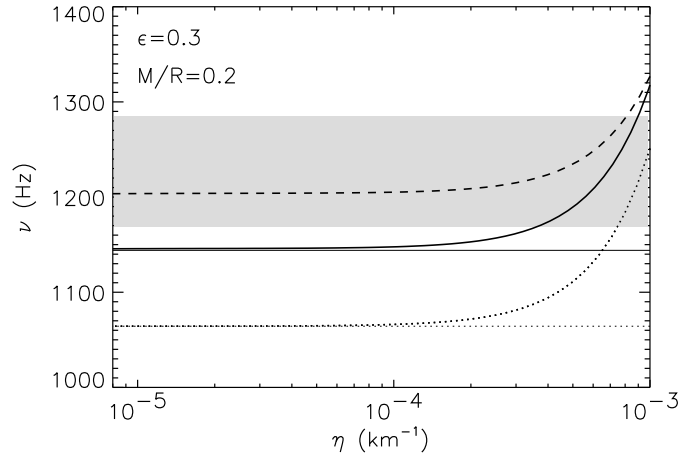
$$\tau = \frac{\rho R^2}{(l-1)(2l+1)\eta}. \quad (9)$$

For our model  $\tau = 3 \times 10^{-8}/\eta$ , with  $\eta$  in  $\text{km}^{-1}$  and  $\tau$  in s. This estimate is also shown in Fig. 2 (thin solid line) and it is, surprisingly, in better agreement with the GR results than with the Newtonian ones.

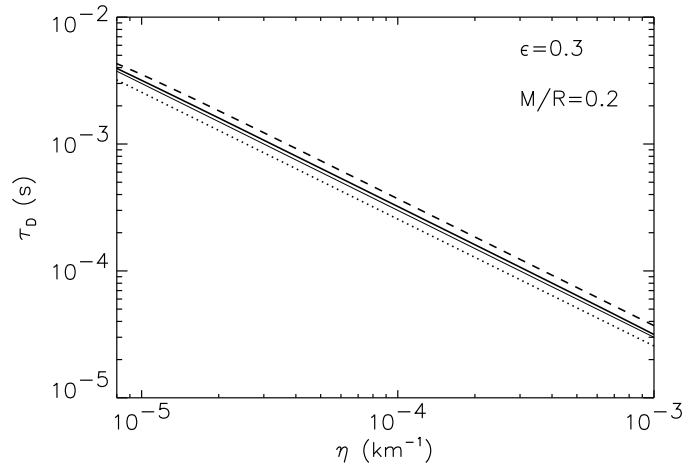
### Polytropic Stars

Here we consider polytropic stars. For these models the  $r$ -mode was found to disappear for  $n \geq 0.8$  [8, 9]. Only for very compact stars  $n < 0.8$ , the  $r$ -mode frequency lies outside the continuous spectrum and could be found. We have considered a polytropic model with  $n = 1$ , and the same compactness and rotation parameter as in previous section ( $M/R = 0.2, \varepsilon = 0.3$ ). This model has mass  $M = 1.74 M_\odot$  and  $R = 12.86 \text{ km}$ . In the inviscid case we do not find the  $r$ -mode because it lies inside the continuous spectrum, consistently with the results of [8]. In Fig. 3 we show the results obtained when viscosity is included in the calculation. In the Newtonian case (dotted line), the inviscid limit is nicely recovered as before. As for the Cowling (dashed line) and RG (solid line) calculations, we can follow the  $r$ -mode inside the continuous spectrum until convergence to the inviscid limit is reached. The corresponding damping times are shown in Fig. 4. The Relativistic damping time lies between the Newtonian and the Cowling calculation, and they agree within a factor 2. For comparison we also include the estimate given by Eq. (9) using the average density of the star, which overestimates the damping time of about 50 %. Notice that using the average density in Eq. (9), the damping time (for models with constant  $\eta$ ) depends only on  $\rho R^2 \propto M/R$ , therefore it gives the same result for stars with the same compactness.

In Fig. 5 we show the behaviour of the real part of the  $r$ -mode frequency ( $\sigma$ ) as a function of the polytropic index  $n$  for models with compactness  $M/R = 0.2$ , period of 1 ms, and a shear viscosity  $\eta = 10^{-6} \text{ km}^{-1}$ . The period and compactness have been chosen to allow for direct comparison with the results of [8] who could not find the  $r$ -modes for  $n$  larger than a certain value, when the real part of the frequency reached the continuous spectrum. By introducing a small amount of viscosity, the frequency can be calculated for all polytropic indexes, or for any other stellar model,



**FIGURE 1.** The  $r$ -mode frequency as a function of the shear viscosity parameter  $\eta$  for a uniform density model with  $M/R = 0.2$  and rotational parameter  $\epsilon = 0.3$ . The Newtonian, Cowling and GR results are shown, respectively, with dotted, dashed, and solid lines. The thin horizontal lines indicate the corresponding Newtonian and GR inviscid limits, while the continuous spectrum is the shadowed region.

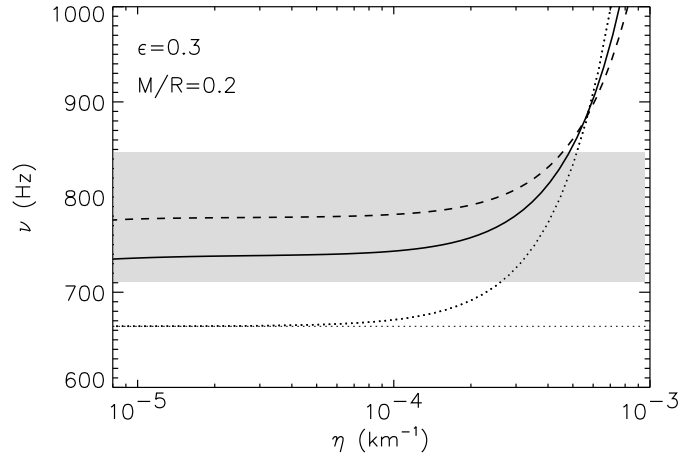


**FIGURE 2.** The  $r$ -mode damping time as a function of the shear viscosity parameter  $\eta$  for a uniform density model with  $M/R = 0.2$  and rotational parameter  $\epsilon = 0.3$ . The Newtonian, Cowling and GR results are shown, respectively, with dotted, dashed, and solid lines. The thin solid line shows the simple estimate obtained using Eq. (9).

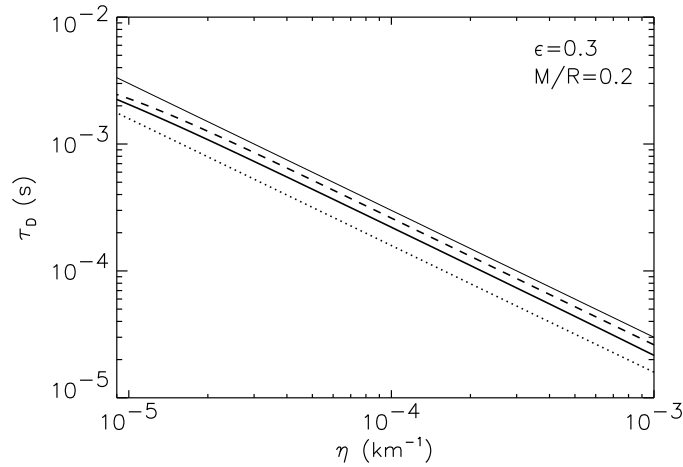
even if we stay at the most basic level of approximation: first order in the rotation parameter and neglecting the coupling between the axial and polar parts. Note that when the mode lays outside the continuous spectrum we obtain results very similar to [8].

## Realistic Neutron Stars

Here we consider stellar models constructed with realistic EOSs and realistic viscosity profiles. At low density (below  $10^{12} \text{ g/cm}^3$ ) we use the BPS [23] equation of state, while for the inner crust  $10^{12} < \rho < 10^{14} \text{ g/cm}^3$  we employ the SLy4 EOS [24]. At high density ( $\rho > 10^{14} \text{ g/cm}^3$ ) we have considered two different EOSs of neutron star matter



**FIGURE 3.** The  $r$ -mode frequency as a function of the shear viscosity parameter  $\eta$  for a polytropic star with  $n = 1$ ,  $M/R = 0.2$  and rotational parameter  $\epsilon = 0.3$ . The Newtonian, Cowling and GR results are shown, respectively, with dotted, dashed, and solid lines. The horizontal dotted line indicates the corresponding Newtonian inviscid limit, while the continuous spectrum is the shadowed region.

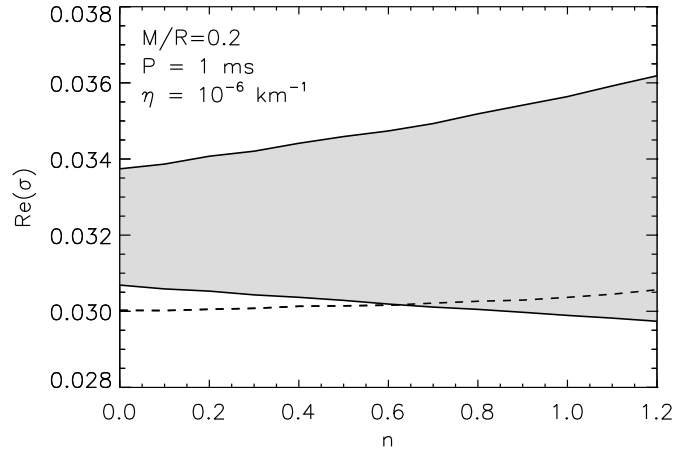


**FIGURE 4.** The  $r$ -mode damping time as a function of the shear viscosity parameter  $\eta$  for a polytropic star with  $n = 1$ ,  $M/R = 0.2$  and rotational parameter  $\epsilon = 0.3$ . The Newtonian, Cowling and GR results are shown, respectively, with dotted, dashed, and solid lines. The thin solid line shows the simple estimate obtained using Eq. (9).

representative of the two different approaches commonly found in the literature: potential models and relativistic field theoretical models. As a potential model, we have chosen the EOS of Akmal-Pandharipande-Ravenhall ([25], hereafter APR). As an example of the mean field solution to a relativistic Walecka-type Lagrangian we have used the parametrization usually known as GM3 [26].

If the compactness is  $M/R = 0.2$ , APR gives a mass of  $1.53 M_{\odot}$  and a radius of 11.3 km, while GM3 gives  $M = 1.72 M_{\odot}$  and  $R = 12.7$  km. For a polytropic EOS with  $n = 1$  the corresponding mass and radius for the same compactness is  $M = 1.74 M_{\odot}$  and  $R = 12.8$  km. For cold neutron stars below  $10^9$  K, neutrons in the inner core become superfluid and the dominant contribution to the shear viscosity is electron-electron scattering (see e.g. the review [27]); in this regime  $\eta$  can be written as

$$\eta_{ee} = 6 \times 10^{18} \rho_{15}^2 T_9^{-2} \text{ g/cm/s} = 1.48 \times 10^{-15} \rho_{15}^2 T_9^{-2} \text{ km}^{-1} \quad (10)$$



**FIGURE 5.** Behaviour of the real part of the  $r$ -mode frequency ( $\sigma$ ) as a function of the polytropic index  $n$  for models with a shear viscosity parameter  $\eta = 10^{-6} \text{ km}^{-1}$ , compactness  $M/R = 0.2$  and period of 1 ms.

with  $\rho_{15}$  and  $T_9$  being the density and temperature in units of  $10^{15} \text{ g/cm}^3$  and  $10^9 \text{ K}$ . Having this in mind, we have used a viscosity coefficient with a quadratic dependence on density

$$\eta = \eta_0 \left( \frac{\rho}{\rho_0} \right)^2 \text{ km}^{-1}, \quad (11)$$

where  $\rho_0$  is the central density. Since old neutrons stars are nearly isothermal, we have parametrized our results as a function of the constant  $\eta_0$  that includes the temperature dependence.

In Fig. 6 we show the  $r$ -mode frequency as a function of  $\eta_0$ , comparing the three EOSs: polytrope (dots), APR (dashes) and GM3 (solid lines). In all cases the rotation period is  $P = 2 \text{ ms}$ . As we can see in the figure, the frequency is rather insensitive to the particular details of the EOS, provided that the rotation frequency and  $M/R$  are the same. The Newtonian limit depends only on the angular frequency ( $\Omega$ ) and the relativistic correction enters through the frame dragging. Since this correction goes as  $\omega/\Omega \approx I/R^3 \propto M/R$ , the leading order contribution to the frequency is again a function only of the compactness.

In Fig. 7 we show the damping times for the same models as in Fig. 6. For constant density stars, the  $\approx \rho R^2$  dependence of the damping time translates into a  $M/R$  dependence for models with constant  $\eta$ . In realistic, cold ( $T \lesssim 10^9 \text{ K}$ ) neutron stars, the density is not constant and the viscosity is dominated by the electron-electron scattering process (10). Thus, the analytic result (9) underestimates the viscous damping time. An improved calculation for  $n = 1$  polytropes taking into account the density profiles (Andersson & Kokkotas 2001) gives

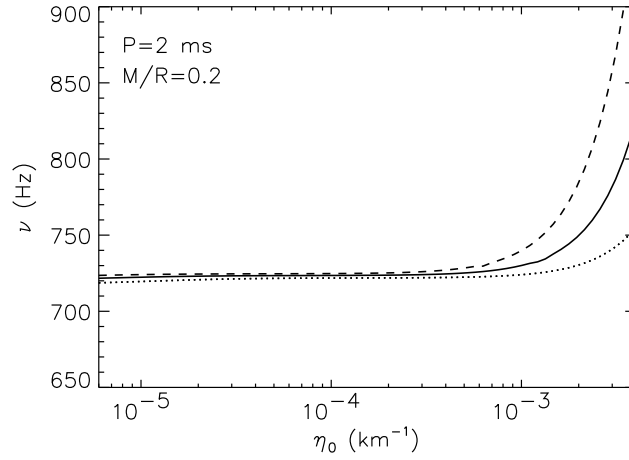
$$\tau = 2.2 \times 10^7 \left( \frac{1.4M_\odot}{M} \right) \left( \frac{R}{10 \text{ km}} \right)^5 T_9^2 \text{ s}. \quad (12)$$

Assuming a general dependence of the viscosity of the form of Eq. (11), the above damping time can be shown to satisfy

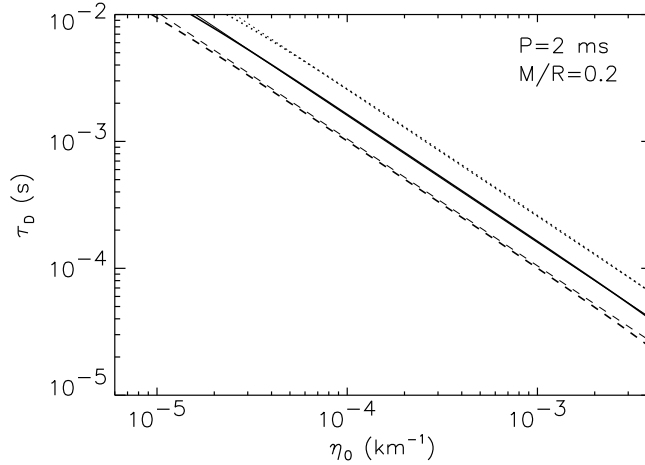
$$\tau = 3.26 \times 10^{-8} \left( \frac{1.4M_\odot}{M} \right) \left( \frac{R}{10 \text{ km}} \right)^5 \left( \frac{\rho_0}{10^{15} \text{ g/cm}^3} \right) \frac{1}{\eta_0} \text{ s}. \quad (13)$$

Our relativistic calculations show approximately the same dependence on mass and radius as Eq. (13), but the damping times are systematically larger of about 60%. We found that a better fit for the realistic neutron stars (APR, GM3) as well as for the  $n = 1$  polytrope is given by

$$\tau = 5.22 \times 10^{-8} \left( \frac{1.4M_\odot}{M} \right) \left( \frac{R}{10 \text{ km}} \right)^5 \left( \frac{\rho_0}{10^{15} \text{ g/cm}^3} \right) \frac{1}{\eta_0} \text{ s}. \quad (14)$$



**FIGURE 6.** Comparison of the  $r$ -mode frequency as a function of the the viscosity coefficient at the center,  $\eta_0$ , for a polytropic star with  $n = 1$  (dotted line), the APR (dashed line) and the GM3 (solid line) equations of state. The compactness parameter in all cases is  $M/R = 0.2$  and the period is 2 ms.



**FIGURE 7.** Comparison of the  $r$ -mode damping time as a function of  $\eta_0$  for a  $n = 1$  polytrope (dotted line), the APR (dashed line) and the GM3 (solid line) equations of state. The compactness parameter in all cases is  $M/R = 0.2$  and the period is 2 ms. The thin lines show the estimates given by Eq. (14), which are difficult to distinguish from the real results.

In Fig. 7 we show, together with the numerical results (thick lines), the results corresponding to the previous fit (thin lines). The good agreement between them is apparent.

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